THE INTERSECTION OF THE POWERS OF THE RADICAL IN NOETHERIAN P.I. RINGS†

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ABSTRACT

Let R be a ring and J its radical. Define $J_1 = \bigcap J^n$, $J_2 = \bigcap J_1^n, \cdots, \cdots$ $J_k = \bigcap J_{k-1}^n \cdots$. It is shown that in a ring R satisfying a polynomial identity and the ascending chain condition on ideals, $J_k = 0$ for some appropriate k.

Unlike the situation in a commutative Noetherian ring, the intersection of the powers of the radical in a noncommutative Noetherian ring may very well be different from (0). In fact, in the first example given of this [4], the ring is a right Noetherian ring satisfying a polynomial identity, that is, a right Noetherian P.I. ring. In that example, however, this intersection at least turned out to be nilpotent. As was later shown [5], if one drops the assumption that the ring be P.I. one can get examples in which this intersection, and higher intersections constructed successively from it, can be rather bizarre.

As we show below in Theorem 1, in the presence of a P.I., and assuming only ascending chain conditions on (two-sided) ideals, although we cannot say that the intersection of the powers of the radical must be 0, we do have some control on its nature. Before going into this in detail, we would like to make a remark or two about the proof of Theorem 1.

The argument makes use of two powerful results proved recently. The first of these, due to M. Artin [2], characterizes Azumaya algebras of rank n^2 over their centers via polynomial identities. The second, due to E. Formanek [3], asserts

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that in the $n \times n$ matrix ring over a field there are polynomials in noncommuting variables which take on nonconstant values in the center.

If R is a ring, let J be the Jacobson radical of R. We define:

$$J_1 = \bigcap_1^\infty J^n,\ J_2 = \bigcap_1^\infty J_1^n, \cdots,\ J_k = \bigcap_1^\infty J_{k-1}^n.$$

Let us recall also that by the *standard identity* of degree m we mean the polynomial $S_m(x_1, \cdots, x_m) = \sum_{\sigma \in \Sigma_m} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ in the noncommuting variables $x_1, \cdots, x_m$, where σ runs over Σ_m, the symmetric group of degree m, and $(-1)^\sigma$ is the signature of σ.

We now prove

THEOREM 1. *Let R be a semi-prime ring satisfying the standard identity of degree $2n$. If R satisfies the ascending chain condition on ideals then $J_n = 0$.*

Before starting the proof we introduce a few notions which we shall need. If R is a prime ring satisfying a polynomial identity, by the *degree* of R we shall mean the degree of the standard identity of lowest degree satisfied by R. If R is a semi-prime P.I. ring, by its degree we mean the maximum of the degrees of R/P, where P is a prime ideal of R.

As was pointed out by several people independently (see [7] for instance), it follows from Posner's theorem and Formanek's result that if R is a prime P.I. ring then its center Z is not (0) and there exist polynomials in noncommuting variables which when evaluated on R give values only in Z, and these values are not a constant. Call a polynomial $f(x_1, \cdots, x_n)$ a *central polynomial* if $f(0,0, \cdots, 0) = 0$, f is not a P.I. for R, but $f(r_1, \cdots, r_n) \in Z$ for all $r_1, \cdots, r_n \in R$. Let $F(R)$ be the set of values taken on by all the central polynomials. $F(R)$ is a subring of R; it is called the *Formanek center* of R.

One simple remark, which is needed in the proof, should be made: if R is a prime ring and P is a prime ideal of R such that $P \supset F(R)$ then R/P has lower degree than R.

PROOF OF THE THEOREM. We claim that in order to prove the theorem it is enough to prove it in case the ring R is prime. For if R is semi-prime, then $0 = \bigcap P$ where P runs over the prime ideals of R. Since R/P is prime and satisfies $S_{2n}(x_1, \cdots, x_{2n})$, if $\bar{J}$ denotes the image of J in R/P, then $\bar{J}_n = 0$, hence $J_n \subset P$. From this and $\bigcap P = 0$ we would have that $J_n = 0$.

So we consider the case that R is a prime ring. We proceed by induction on the degree of R.

If there exists an element $\alpha \in F(R)$, the Formanek center, such that $\alpha \notin J$ then, since R/J is semi-prime and $\alpha + J$ is central in R/J, $\alpha + J$ is not nilpotent, and hence $\alpha^i \notin J$ for any integer i. The localization, $R[\alpha^{-1}]$, of R at α is an Azumaya algebra [6], and $JR[\alpha^{-1}] \neq R[\alpha^{-1}]$. Since $R[\alpha^{-1}]$ is an Azumaya algebra, there is a one-to-one correspondence between the ideals and those of its center Z; moreover, if U is an ideal of $R[\alpha^{-1}]$, then $U = (U \cap Z)R[\alpha^{-1}]$. Since Z is a commutative Noetherian domain, if $J_0 = (JR[\alpha^{-1}]) \cap Z$, then $\bigcap J_0^m = 0$. Because $JR[\alpha^{-1}] = J_0R[\alpha^{-1}]$, and from the correspondence of the ideals of $R[\alpha^{-1}]$ with those of Z, we get that $\bigcap (JR[\alpha^{-1}])^m = 0$, and so $\bigcap J^m = 0$. That is, in this case, $J_1 = 0$.

Therefore we may assume that $F(R) \subset J$. Since $F(R) \subset Z$, the argument given in the commutative case yields that $\bigcap (F(R)R)^m = 0$. Let K in R be the inverse image of $\bar{K}$, the maximal nil ideal of $R/F(R)R$. Since $R/F(R)R$ is P.I. and satisfies the ascending chain condition on ideals, $\bar{K}$ is nilpotent; say $\bar{K}^s \subset F(R)R$; in particular, $K \subset J$.

Since $\bar{K}$ is the intersection of all the prime ideals of $R/F(R)R$, K is the intersection, $K = \bigcap P$, where P runs over all the prime ideals of R which contain $F(R)R$. Thus, all central polynomials of R vanish on R/P for each prime ideal P of R which contains $F(R)$. Hence R/P must be of lower degree than R; thus R/P must satisfy $S_{2(n-1)}(x_1, \cdots, x_{2(n-1)})$. By the induction we have that $J_{n-1} \subset P$ for each such P. But then $J_{n-1} \subset \bigcap P = K$. Since $K^s \subset F(R)R$ and $\bigcap (F(R)R)^m = 0$, we get that $\bigcap J_{n-1}^m = 0$, and so $J_n = 0$ as desired.

Theorem 1 immediately implies

THEOREM 2. *Let R be a ring satisfying the ascending chain condition on ideals. If R satisfies the standard identity of degree $2n$ then $J_{n+1} = 0$ (in fact, J_n is nilpotent).*

PROOF. If N is the maximal nil ideal of R, then N is nilpotent. Moreover, since R/N satisfies the hypothesis of Theorem 1, $J_n \subset N$ follows, and thus the theorem is true.

The next and final theorem is in the same spirit as Theorem 2. Its proof is completely elementary and formal; as a corollary to it we obtain a special case of Theorem 2.

But first a

DEFINITION. A ring is (left) *bounded* if in every prime homomorphic image, left ideals generated by a regular element contain a nonzero two-sided ideal.

THEOREM 3. *Let R be a right Noetherian ring which is left bounded; then $J_t = 0$ for some integer t.*

PROOF. Assume that $J_s \neq 0$ for all s. Let I be an ideal of R maximal with respect to the property of containing no J_s. We claim that I is a prime ideal of R. For if $AB \subset I$ where A and B are ideals of R which properly contain I, then by the choice of I, $J_u \subset A$ and $J_v \subset B$ for some u and v. Thus $J_s \subset A$ and $J_s \subset B$ where $s = \max(u, v)$; hence, $J_s^2 \subset AB \subset I$. But then $J_{s+1} = \bigcap J_s^m$ is in I, contrary to the choice of I. Therefore I is a prime ideal of R.

Hence, to prove the theorem, we may assume that R is a prime ring in which $J_t \neq 0$ for all t, but J_s, for appropriate s, is contained in every nonzero ideal of R.

Now, as an ideal in a prime right Noetherian ring, J must contain a regular element a. By assumption, $Ra \supset T$, $T \neq 0$ a nonzero ideal of R. From its form, $T = Ia$, where I is a left ideal of R. Now $T^2 = IaIa$ and $IaI \subset T \subset Ra$, hence $T^2 \subset Ra^2$. Continuing in this manner we get that $T^n \subset Ra^n$ for every n. Thus $\bigcap T^n \subset \bigcap Ra^n$. If $c \neq 0$ is in $\bigcap T^n$ then $c = r_1 a = r_2 a^2 = \cdots = r_n a^n = \cdots$. Because a is regular we have $r_n R \subset r_{n+1} R$; now the ascending chain of right ideals $r_1 R \subset r_2 R \subset \cdots \subset r_n R \subset \cdots$ terminates at some point, that is, $r_n R = r_{n+1} R$. So $r_{n+1} = r_n b$ for some $b \in R$ but $r_n = r_{n+1} a$, which gives that $r_{n+1} = r_n b = r_{n+1} ab$. Since $ab \in J$ this last relation forces $r_{n+1} = 0$, and so $c = r_{n+1} a^{n+1} = 0$, a contradiction. Thus $\bigcap Ra^n = 0$, and so $\bigcap T^n = 0$. Since $T \neq 0$ is an ideal of R, $J_s \subset T$, but then $J_{s+1} = \bigcap J_s^n \subset \bigcap T^n = 0$. With this contradiction the theorem is proved.

Theorem 3 has an interesting corollary.

COROLLARY. *If R is a right Noetherian P.I. ring then $J_t = 0$ for some t.*

PROOF. By a result of Amitsur [1], a P.I. ring is left bounded. Apply the theorem.

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